Stiefel-Whitney classes

These are characterized by axiomatic properties — the existence & uniqueness will be seen later.

Axioms: For each real vector bundle $\pi: E \to B$ of rank r, $\exists$ cohomology classes $w_i(E) \in H^i(B, \mathbb{Z}/2)$ $i = 0 \dots r$, with $w_0(E) = 1 \in H^0(B, \mathbb{Z}/2)$, s.t.

ith Stiefel-Whitney class of E

1) Naturality:
$$\begin{array}{cc} E' & E \\ \downarrow & \downarrow \pi \\ f: A & \to B \end{array} \qquad E' \cong f^* E \Rightarrow w_i(E') = f^* w_i(E)$$

2) Whitney sum: $E, F \to B \Rightarrow w_k(E \oplus F) = \sum_{i=0}^{k} w_i(E) \cup w_{k-i}(F)$

cup-product.

eg. $w_1(E \oplus F) = w_1(E) + w_1(F)$
$w_2 = w_2(E) + w_1(E) \cup w_1(F) + w_2(F)$.

(recall in simplicial cohomology,
$$(\alpha \cup \beta)(\underbrace{[v_0 \dots v_k]}_{k\text{-simplex}}) := \alpha(\underbrace{[v_0 \dots v_i]}_{i\text{-simplex}})\, \beta(\underbrace{[v_i \dots v_k]}_{(k-i)})$$
)

3) Nontriviality: for the tautological line bundle $\tau \to \mathbb{R}P^1$, $w_1(\tau) \neq 0$.

Immediate properties:

- $E \simeq E'$ isomorphic $\Rightarrow w_i(E) = w_i(E')$.
- $E \to B$ trivial $\Rightarrow w_i(E) = 0 \;\forall i > 0$ (since can write E as pullback $B \to pt$, $\mathbb{R}^r \downarrow pt$)
- E trivial $\Rightarrow w_i(E \oplus E') = w_i(E')$
- E Eucl. rank r, with k pointwise linearly indept sections $\Rightarrow w_i(E) = 0$ for $i > r-k$.
  (write E as Whitney sum of trivial rk k bundle and rank $r-k$ complement)

For convenience, def. $H^*(B, \mathbb{Z}/2) := \prod_{i \geq 0} H^i(B, \mathbb{Z}/2)$ total cohomology ring
$\ni a_0 + a_1 + a_2 + \dots$, $a_i \in H^i(B, \mathbb{Z}/2)$.

$\cup$ defines a graded commutative ring structure.

and let the total Stiefel-Whitney class $w(E) := 1 + w_1(E) + \dots + w_r(E) \in H^*(B, \mathbb{Z}/2)$.
Then $w(E \oplus F) = w(E)\, w(F)$

Lemma: elements of the form $a = 1 + a_1 + a_2 + \dots \in H^*(B, \mathbb{Z}/2)$ form a group under mult.

Pf.: $a^{-1} = 1 + \bar{a}_1 + \bar{a}_2 + \dots$ solve inductively for $\bar{a}_k$ by looking at deg k part of $a \cdot a^{-1}$.

namely $a_k + a_{k-1}\bar{a}_1 + \ldots + a_1 \bar{a}_{k-1} + \bar{a}_k = 0$ determines $\bar{a}_k$ once $\bar{a}_1 \ldots \bar{a}_{k-1}$ known.

(in fact $\bar{a}_1 = a_1$, $\bar{a}_2 = a_1^2 + a_2$, ...)

Corollary: $w(E) = w(F)^{-1} w(E \oplus F)$. In particular if $E \oplus F$ is trivial then $w(E) = w(F)^{-1}$.

Eg. this applies to: $M \subset \mathbb{R}^n$ smooth submanifold $\Rightarrow$ $TM \oplus NM \cong \underline{\mathbb{R}}^n_{|M}$
or M immersed into $\mathbb{R}^n$

so $W(NM) = W(TM)^{-1}$.

Example 1: $S^n \subset \mathbb{R}^{n+1}$ has trivial normal bundle (nonvanishing section $s(x) = x$)

so $w(TS^n) = w(NS^n)^{-1} = 1$.

Stiefel-Whitney classes don't detect the nontriviality of TS^2.

[note: TS^1, TS^3 are trivial! $T_xS^1 \ni ix$, $T_xS^3 \ni ix, jx, kx$.]
S^1, S^3, S^7 are parallelizable — 3 indept sections

Recall: the cohomology of $\mathbb{RP}^n$ is $H^i(\mathbb{RP}^n, \mathbb{Z}/2) = \mathbb{Z}/2$ $\forall\, 0 \le i \le n$

(cellular chain complex $C_{i+1} = \mathbb{Z} \xrightarrow[0 \text{ or } 2]{\partial} C_i = \mathbb{Z}$

and on $\mathrm{Hom}(\cdot, \mathbb{Z}/2)$ this becomes $C^i = \mathbb{Z}/2 \xrightarrow{0} C^{i+1} = \mathbb{Z}/2$)

and as ring, denoting by $a \in H^1(\mathbb{RP}^n, \mathbb{Z}/2)$ the nonzero elt,

$a^k =$ generator of $H^k(\mathbb{RP}^n, \mathbb{Z}/2)$ $\forall k$, ie. $H^*(\mathbb{RP}^n, \mathbb{Z}/2) \simeq \mathbb{Z}/2[a]/a^{n+1}$.

(Pf: induction on n. If true for $\mathbb{RP}^n$ then:

incl. $\mathbb{RP}^n \underset{i}{\hookrightarrow} \mathbb{RP}^{n+1}$: i^* induces $\simeq$ on $H^0, \ldots, H^n$, maps $a \mapsto a$, and preserves multiplicative structure.

so true up to a^n. Moreover Poincaré duality (over $\mathbb{Z}/2$):
$\mathbb{RP}^{n+1}$ closed manifold,

$H^1(\mathbb{RP}^{n+1}, \mathbb{Z}/2) \times H^n(\mathbb{RP}^{n+1}, \mathbb{Z}/2) \xrightarrow{\cup} H^{n+1} = \mathbb{Z}/2$ nondegenerate

$\Rightarrow a^n \cup a = a^{n+1} \neq 0$ ▲).

Example 2: $\tau \to \mathbb{RP}^n$ tautological bundle $\Rightarrow$ $w(\tau) = 1 + a$.

Pf: pullback by inclusion $\mathbb{RP}^1 \underset{i}{\hookrightarrow} \mathbb{RP}^n$: $i^*\tau = \tau_{|\mathbb{RP}^1} =$ taut. bundle of $\mathbb{RP}^1$.

so $i^* w_1(\tau) = w_1(i^*\tau) \neq 0$ $\Rightarrow$ $w_1(\tau) \neq 0$, so $= a$. ▲

Example 3: tangent bundle of $\mathbb{RP}^n$,

Lemma: $T\mathbb{RP}^n \simeq \mathrm{Hom}(\tau, \tau^\perp)$. ($\tau^\perp$ orth. complement of $\tau \subset \underline{\mathbb{R}}^{n+1}$)

Pf: $TS^n = \{(x,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,/\, |x| = 1,\ x\cdot v = 0\}$

$\downarrow D\pi$

$T\mathbb{RP}^n$ = same up to $(x,v) \sim (-x,-v)$ (antipodal involution)

Such a pair $(x,v) \Leftrightarrow$ linear mapping $\ell : L = \mathbb{R}.x \to L^\perp$, $tx \mapsto tv$

and this gives $T_{[x]}\mathbb{RP}^n \underset{\text{canonically}}{\simeq} \mathrm{Hom}(L, L^\perp)$

Corollary: $T\mathbb{RP}^n \oplus \underline{\mathbb{R}} \simeq \mathrm{Hom}(\tau,\tau^\perp) \oplus \mathrm{Hom}(\tau,\tau)$ ← trivial (nonvanish. sec. id)

$= \mathrm{Hom}(\tau, \underline{\mathbb{R}}^{n+1}) = \tau^* \oplus \dots \oplus \tau^* \simeq \tau \oplus \dots \oplus \tau.$

using Eucl. metric

Hence $w(T\mathbb{RP}^n) = w(T\mathbb{RP}^n \oplus \underline{\mathbb{R}}) = w(\tau)^{n+1} = \underline{(1+a)^{n+1}}.$

(note $a^{n+1} = 0$)

Ex: $w(\mathbb{RP}^1) = 1$

$w(\mathbb{RP}^2) = 1 + a + a^2$

$w(\mathbb{RP}^3) = 1$

$w(\mathbb{RP}^4) = 1 + a + a^4.$

Corollary: (Stiefel) $\mathbb{RP}^n$ is parallelizable $\Rightarrow n = 2^k - 1 \in \{1, 3, 7, 15 \dots\}$

(only case where $\binom{n+1}{i}$ all even $\forall 1 \le i \le n$).

[$\mathbb{RP}^{1,3,7}$ parallelizable using $\mathbb{C}, \mathbb{H}, \mathbb{O}$; others actually aren't.].

Application: If M^n admits an immersion into $\mathbb{R}^{n+k}$ then $w(TM)^{-1} = 1 + \bar{w}_1 + \bar{w}_2 + \dots$ has $\bar{w}_i = 0$ for $i > k$.

(since $w(NM) = w(TM)^{-1}$ but $rk(NM) = k$).

Ex: for $\mathbb{RP}^4$, $w(\mathbb{RP}^4)^{-1} = 1 + a + a^2 + a^3$ so $\mathbb{RP}^4 \not\hookrightarrow \mathbb{R}^6$

but Whitney's thm $\Rightarrow \mathbb{RP}^4 \hookrightarrow \mathbb{R}^{2\cdot 4 - 1 = 7}$: 7 is optimal.

(similarly for $n = 2^m$, $\mathbb{RP}^n \not\hookrightarrow \mathbb{R}^{2n-2}$, though Whitney $\Rightarrow \hookrightarrow \mathbb{R}^{2n-1}$).

Stiefel-Whitney numbers and cobordisms:

- to get numerical invts of closed manifolds, rather than cohomology classes: integrate $w_i(TM)$ against fundamental class $[M] \in H_n(M, \mathbb{Z}/2)$.

Stiefel-Whitney numbers $:= \langle w_1(TM)^{r_1} \dots w_n(TM)^{r_n}, [M] \rangle \in \mathbb{Z}/2 \quad \forall r_i$ st. $\sum i r_i = n$.

Ex: for n odd, $n=2k-1$, $w(T\mathbb{RP}^n) = (1+a)^{2k} = (1+a^2)^k$
so all odd w_i's are zero $\Rightarrow$ all Stiefel-Whitney #'s are zero
for n even, $w_n(T\mathbb{RP}^n) = (n+1)\,a^n = a^n \neq 0$, and $w_1^{\,n} = ((n+1)a)^n = a^n \neq 0$

Why we care? Let M be a closed smooth n-manifold, not necess. connected.

Thm (Pontrjagin $\Rightarrow$)
Thom $\Leftarrow$
$\exists$ B smooth compact $(n+1)$-mfld with boundary $\partial B = M$
$\Leftrightarrow$ all the Stiefel-Whitney numbers of M are zero.

Pf of $\Rightarrow$ ($\Leftarrow$ is much harder).

Let $[B] \in H_{n+1}(B, M; \mathbb{Z}/2)$ fundamental class, then $[M] = \partial([B])$ under $\partial : H_{n+1}(B,M) \to H_n(M)$.

Now TB well-defined (even at boundary) and $TB_{|M} \simeq TM \oplus \underset{\uparrow}{\underline{\mathbb{R}}}$

outward normal v.f. to ∂B

so $w_i(TB)_{|M} = w_i(TM)$.

Hence any polynomial $\mathcal{P}$ in Stiefel-Whitney classes of TM is in the image of $H^n(B) \xrightarrow{i^*} H^n(M)$, hence $\mathcal{P} \in \ker(\delta : H^n(M) \to H^{n+1}(B,M))$.

Thus $\langle \mathcal{P}, [M] \rangle = \langle \mathcal{P}, \partial([B]) \rangle = \langle \delta(\mathcal{P}), [B] \rangle = 0$. $\blacktriangle$

Corollary: M_1, M_2 smooth closed n-mflds are unorientedly smoothly <u>cobordant</u> (i.e. $\exists$ smooth compact manifold B^{n+1}, not necess. orientable, s.t. $\partial B = M_1 \sqcup M_2$) iff all of their Stiefel-Whitney numbers are equal.